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An analysis of advective diffusion in branching channels

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Solutions to the steady advection–diffusion equation in a branching channel are obtained for both uniform and spatially varying flow fields and for two channel geometries. An interesting feature of the solutions is that anisotropy of the dispersion coefficients in the direction of the streamlines may be accounted for. The analysis reveals that mixing is confined to a distance, b^2U/π^2K_N, downstream of the junction in the advection-dominated case and a distance, K_S/U, upstream in the diffusion-dominated situation, K_S and K_N being the diffusivities along and across the flow respectively, U the characteristic velocity of the flow, and b the breadth of channel downstream of the junction.

1. Introduction

The field of concentration arising from the mixing of two fluids of differing flow and concentrations at a junction of two channels is of interest in a variety of applications. Sayre (1973) in his consideration of the problem of the mixing of two rivers assumed that the concentration field was completely unmixed at the point of confluence. On the other hand, in a treatment of dispersion in a network of tubes, Ultman & Blatman (1977) assumed complete lateral mixing of the tracer across each of the channels in a branching system. It is the purpose of the present work to show under what circumstances such assumptions are valid. Also it will be shown that an exact solution for the concentration field can be constructed and how, under some modifications, it can be applied to more general conditions of flow and geometry.

2. The diffusion equation model

The method of study that will be adopted is the diffusion equation approach for the reasons stated by Hunt & Mulhearn (1973). The advantage of this approach over the statistical analysis is that the effects of the boundary condition of no normal material flux is more readily calculable than by statistical methods and that effects of the straining of the flow by the convergence or divergence of the streamlines may be accounted for. Finally, since it is assumed that the concentration is either uniform or fairly well mixed upstream, errors introduced by the neglect of the effect of small-scale influences on the diffusion process can be neglected. The two-dimensional diffusion equation with a constant molecular or eddy diffusivity of K_S in the direction

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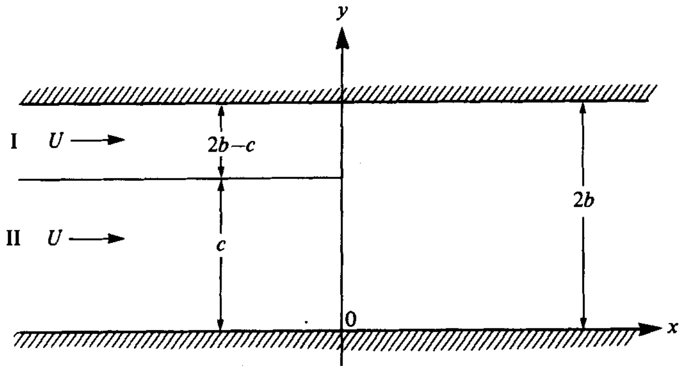

FIGURE 1. Configuration for flow in two channels of widths c and $2b-c$, which unite at $x = 0$, $y = c$.

of the fluid flow and of K_N in the direction normal to the streamlines may be written as

$$U\frac{\partial C}{\partial x} = K_S\frac{\partial^2 C}{\partial x^2} + K_N\frac{\partial^2 C}{\partial y^2}, \tag{2.1}$$

where U is the mean velocity, and x is taken as parallel to the flow and y normal to the flow.

Suppose that flow is confined to two parallel channels which unite at $x = 0$ (see figure 1) and that the speed of the flow, U, in each of the separate channels is equal.

The boundary conditions on the channel walls are assumed to be $\partial C/\partial y = 0$, $y = 0, 2b$, $-\infty < x < \infty$, and $\partial C/\partial y = 0$, $y = c$, $-\infty < x \leqslant 0$. The quantities C and $\partial C/\partial y$ are continuous on $y = c$, $0 \leqslant x < \infty$.

The concentrations at $x \to -\infty$ are assumed to be C_{II} in $0 \leqslant y \leqslant c$ and C_I in $c < y \leqslant 2b$.

3. Analysis

3.1. *Exact*

The number of variables in the solution may be reduced by introducing the horizontal length scales

$$L_x = K_S/U, \quad L_y = (K_S K_N)^{\frac{1}{2}}/U;$$

then

$$x' = x/L_x, \quad b' = b/L_y,$$
$$y' = y/L_y, \quad c' = c/L_y.$$

Introducing the dimensionless variables x', y' and dropping the primes, the diffusion equation may be written as

$$\frac{\partial C}{\partial x} = \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}.$$

In region I, we define $C'(x,y)$ such that $C' = 0$ as $x \to -\infty$ and similarly, in region II, $C'(x,y) = 0$ as $x \to -\infty$. Then

$$C' = C - C_I, \quad c \leqslant y \leqslant 2b, \tag{3.1}$$
$$C' = C - C_{II}, \quad 0 \leqslant y \leqslant c. \tag{3.2}$$

Thus C' will be discontinuous along $y = c$ for $-\infty < x < \infty$ but we require that $\partial C'/\partial y$ be continuous on $y = c$ for $0 \leqslant x < \infty$.

A simplification is obtained through the introduction of the variable Ψ,

$$\Psi = C'(x,y)\exp(-\tfrac{1}{2}x). \tag{3.3}$$

Then Ψ satisfies

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}-\tfrac{1}{4}\Psi = 0. \tag{3.4}$$

When a Fourier transform is applied to (3.4),

$$\Phi(y,\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,y)\,e^{i\alpha x}\,dx,$$

and when the condition that $\partial\Phi/\partial y = 0$, $y = 0, 2b$ is employed we have

$$\begin{aligned}\Phi(y) &= A(\alpha)\cosh\gamma y \quad \text{for} \quad 0 \leqslant y \leqslant c,\\ &= B(\alpha)\cosh\gamma(2b-y) \quad \text{for} \quad c \leqslant y \leqslant 2b,\end{aligned}$$

where $\gamma = (\alpha^2+\frac{1}{4})^{\frac{1}{2}}$. The functions A and B may be determined from the Wiener–Hopf procedure. Following this technique we introduce the Fourier transforms

$$\Phi(\alpha,y) = \Phi_+(\alpha,y)+\Phi_-(\alpha,y),$$

where

$$\Phi_+(\alpha,y) = (2\pi)^{-\frac{1}{2}}\int_0^{\infty}\Psi\,e^{i\alpha x}\,dx$$

and

$$\Phi_-(\alpha,y) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{0}\Psi\,e^{i\alpha x}\,dx.$$

Then on $y = c$ we have that $\Phi_\pm(c)$ are discontinuous, $(\partial\Phi_-/\partial y)$ $(c\pm 0) = 0$ and that $\partial\Phi_+/\partial y$ is continuous. In addition,

$$\begin{aligned}\Phi_+(c+0)-\Phi_+(c-0) &= \frac{C_{\mathrm{II}}-C_{\mathrm{I}}}{(2\pi)^{\frac{1}{2}}}\int_0^{\infty}e^{ix(\alpha+\frac{1}{2}i)}\,dx\\ &= \frac{C_{\mathrm{II}}-C_{\mathrm{I}}}{(2\pi)^{\frac{1}{2}}(\frac{1}{2}-i\alpha)}.\end{aligned} \tag{3.5}$$

On $y = c$ we may write

$$\Phi_+(c+0)+\Phi_-(c+0) = B(\alpha)\cosh\gamma(2b-c), \tag{3.6}$$

$$\Phi_+(c-0)+\Phi_-(c-0) = A(\alpha)\cosh\gamma c, \tag{3.7}$$

$$\begin{aligned}\frac{\partial\Phi_\pm}{\partial y}(c) &= \gamma A(\alpha)\sinh\gamma c\\ &= -\gamma B(\alpha)\sinh\gamma(2b-c).\end{aligned} \tag{3.8}$$

We define $D_- = \Phi_-(c+0)-\Phi_-(c-0)$.

By subtracting (3.7) from (3.6), eliminating B and A from (3.8) and substituting from (3.5),

$$D_- = -\frac{\partial\Phi_+(c)}{\partial y}\,\frac{\sinh\gamma 2b}{\gamma\sinh\gamma c\sinh\gamma(2b-c)}+\frac{(C_{\mathrm{I}}-C_{\mathrm{II}})\,i}{(2\pi)^{\frac{1}{2}}(\alpha+\frac{1}{2}i)}. \tag{3.9}$$

Define

$$K(\alpha) = \frac{c(2b-c)\,\gamma\sinh\gamma 2b}{2b\sinh\gamma c\sinh\gamma(2b-c)} = K_+(\alpha)\,K_-(\alpha),$$

where $|K_+|$ and $|K_-|\sim|\alpha|^{\frac{1}{2}}$ as $\alpha\to\infty$ in appropriate half-planes. The factorization of $K(\alpha)$ is given in the appendix.

The crucial step is the rearrangement of (3.9) so that there is a common domain containing the inversion path. Rewriting (3.9) as

$$\frac{(\alpha-\frac{1}{2}i)\,D_-}{K_-(\alpha)}-\frac{(C_{\mathrm{I}}-C_{\mathrm{II}})\,i}{(2\pi)^{\frac{1}{2}}(\alpha+\frac{1}{2}i)}\left(\frac{\alpha-\frac{1}{2}i}{K_-(\alpha)}+\frac{i}{K_-(-\frac{1}{2}i)}\right)$$
$$=-\frac{2b}{c(2b-c)}\frac{\partial\Phi_+(c)}{\partial y}\frac{K_+(\alpha)}{(\alpha+\frac{1}{2}i)}+\frac{C_{\mathrm{I}}-C_{\mathrm{II}}}{(2\pi)^{\frac{1}{2}}(\alpha+\frac{1}{2}i)\,K_-(-\frac{1}{2}i)}.$$

By the application of the Wiener–Hopf technique each side of the above equation is identically zero, see Noble (1958).

We now have
$$\frac{\partial\Phi_+(c)}{\partial y}=\frac{c(2b-c)}{2b}\frac{(C_{\mathrm{I}}-C_{\mathrm{II}})}{(2\pi)^{\frac{1}{2}}K_+(\alpha)\,K_-(-\frac{1}{2}i)}.$$

Thus for $-\infty < x \leqslant 0$,

$$A(\alpha)=\frac{c(2b-c)\,(C_{\mathrm{I}}-C_{\mathrm{II}})}{2b(2\pi)^{\frac{1}{2}}\gamma K_+(\alpha)\,K_+(\frac{1}{2}i)\sinh\gamma c} \tag{3.10}$$

$$B(\alpha)=-\frac{c(2b-c)\,(C_{\mathrm{I}}-C_{\mathrm{II}})}{2b(2\pi)^{\frac{1}{2}}\gamma K_+(\alpha)\,K_+(\frac{1}{2}i)\sinh\gamma(2b-c)}; \tag{3.11}$$

and, for $0 \leqslant x < \infty$ and from the definition of $K(\alpha)$.

$$A(\alpha)=\frac{(C_{\mathrm{I}}-C_{\mathrm{II}})\,K_-(\alpha)\sinh\gamma(2b-c)}{(2\pi)^{\frac{1}{2}}K_+(\frac{1}{2}i)\,\gamma^2\sinh\gamma 2b}, \tag{3.12}$$

$$B(\alpha)=-\frac{(C_{\mathrm{I}}-C_{\mathrm{II}})\,K_-(\alpha)\sinh\gamma c}{(2\pi)^{\frac{1}{2}}K_+(\frac{1}{2}i)\,\gamma^2\sinh\gamma 2b}. \tag{3.13}$$

The distribution of Ψ at any point is given by

$$\Psi=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty+\frac{1}{2}i}^{\infty+\frac{1}{2}i}A(\alpha)\cosh\gamma y\,e^{-i\alpha x}\,d\alpha \quad (0\leqslant y\leqslant c), \tag{3.14}$$

$$\Psi=\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty+\frac{1}{2}i}^{\infty+\frac{1}{2}i}B(\alpha)\cosh\gamma(2b-y)\,e^{-i\alpha x}\,d\alpha \quad (c\leqslant y\leqslant 2b). \tag{3.15}$$

These integrals are readily determined by contour integration since the only singularities are simple poles.

Finally, we have for the mean field of concentration and after applying (3.1), (3.2), (3.3), for $-\infty < x \leqslant 0$ and $0 \leqslant y < c$,

$$C(x,y)=(C_{\mathrm{I}}-C_{\mathrm{II}})\frac{2b-c}{2b}\left[\frac{\exp(x)}{K_+^2(\frac{1}{2}i)}+\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi y/c)\exp[(\gamma_n+\frac{1}{2})x]}{\gamma_n K_+(i\gamma_n)\,K_+(\frac{1}{2}i)}\right]+C_{\mathrm{II}}, \tag{3.16a}$$

where
$$\gamma_n=\left(\left(\frac{n\pi}{c}\right)^2+\tfrac{1}{4}\right)^{\frac{1}{2}};$$

and, in $c \leqslant y \leqslant 2b$,

$$C(x,y)=-(C_{\mathrm{I}}-C_{\mathrm{II}})\frac{c}{2b}\left[\frac{\exp(x)}{K_+^2(\frac{1}{2}i)}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{(-1)^n\cos[n\pi(2b-y)/(2b-c)]\exp((\gamma_n+\frac{1}{2})x)}{\gamma_n K_+(i\gamma_n)\,K_+(\frac{1}{2}i)}\right]+C_{\mathrm{I}}, \tag{3.16b}$$

where
$$\gamma_n=\left(\left(\frac{n\pi}{2b-c}\right)^2+\tfrac{1}{4}\right)^{\frac{1}{2}}.$$

For $0 \leqslant x < \infty$ and $0 \leqslant y \leqslant c$,

$$C(x,y) = C_{\mathrm{INF}} + (C_{\mathrm{I}} - C_{\mathrm{II}}) \sum_{n=1}^{\infty} \frac{(-1)^n K_+(i\gamma_n) \sin(n(2b-c)\pi/2b)}{n\pi\gamma_n K_+(\frac{1}{2}i)} \times \cos\left(\frac{n\pi y}{2b}\right) \exp\left(\left(\tfrac{1}{2} - \gamma_n\right)x\right); \quad (3.16c)$$

and in $c \leqslant y \leqslant 2b$,

$$C(x,y) = C_{\mathrm{INF}} - (C_{\mathrm{I}} - C_{\mathrm{II}}) \sum_{n=1}^{\infty} \frac{(-1)^n K_+(i\gamma_n) \sin(cn\pi/2b)}{n\pi\gamma_n K_+(\frac{1}{2}i)} \times \cos\frac{n\pi}{2b}(2b-y) \exp\left(\left(\tfrac{1}{2} - \gamma_n\right)x\right), \quad (3.16d)$$

where $$C_{\mathrm{INF}} = ((2b-c)\, C_{\mathrm{I}} + cC_{\mathrm{II}})/2b,$$

and $$\gamma_n = \left(\left(\frac{n\pi}{2b}\right)^2 + \tfrac{1}{4}\right)^{\frac{1}{2}} \quad \text{for} \quad 0 \leqslant x < \infty.$$

3.2. *Asymptotic solution for large channel width*

In either the advection-dominated case or in the case of large channel width we may consider the limit as $b \to \infty$ of, say, integral (3.14) and (3.10)

$$C(x,y) = \frac{(C_{\mathrm{I}} - C_{\mathrm{II}})}{2\pi} \int_{-\infty+\frac{1}{2}i}^{\infty+\frac{1}{2}i} \frac{c(2b-c)\exp\left[(-\gamma c + \gamma(y^1+c) - i\alpha x + \frac{1}{2})\right]}{2b\gamma[c(2b-c)/b]^{\frac{1}{2}}\left(\frac{1}{2} - i\alpha\right)^{\frac{1}{2}}[c(2b-c)/b]^{\frac{1}{2}}} d\alpha + C_{\mathrm{II}},$$

where $$y^1 = y - c;$$

the approximation for $K_+(\alpha)$ (appendix), and the decaying exponential form for $\cosh \gamma y$ have been used. On writing $\alpha^1 = \alpha + \frac{1}{2}i$,

$$\begin{aligned} C(x,y) &= \frac{(C_{\mathrm{I}} - C_{\mathrm{II}})\, e^{\frac{1}{4}i\pi}}{4\pi} \int_{-\infty}^{\infty} \frac{\exp\left[y^1(\alpha^{12} - i\alpha^1)^{\frac{1}{2}} - i\alpha^1 x\right]}{(\alpha^{12} - i\alpha^1)^{\frac{1}{2}}(\alpha^1)^{\frac{1}{2}}} d\alpha^1 + C_{\mathrm{II}} \\ &= \tfrac{1}{2}(C_{\mathrm{I}} - C_{\mathrm{II}})(1 - \operatorname{erf}(\eta)) + C_{\mathrm{II}} \\ &= \tfrac{1}{2}(C_{\mathrm{I}} + C_{\mathrm{II}}) + \tfrac{1}{2}(C_{\mathrm{II}} - C_{\mathrm{I}}) \operatorname{erf}(\eta) \end{aligned} \quad (3.17a)$$

in $y^1 \leqslant 0$ where $\eta = \operatorname{Im}(x - iy^1)^{\frac{1}{2}}$.

Similarly for $y^1 \geqslant 0$ the asymptotic form reduces to

$$C(x,y) = \tfrac{1}{2}(C_{\mathrm{I}} + C_{\mathrm{II}}) + \tfrac{1}{2}(C_{\mathrm{I}} - C_{\mathrm{II}}) \operatorname{erf}(\eta), \quad (3.17b)$$

where $\eta = \operatorname{Im}(x + iy^1)^{\frac{1}{2}}$.

This relatively simple form of the solution may have practical application in the vicinity of the confluence of the two channels.

4. Non-uniform velocity fields

Hunt & Mulhearn (1973) have pointed out that for diffusion problems involving potential flows the diffusion equation can be solved readily in complicated flow fields by the transformation of Boussinesq.

In the more general case of anisotropic diffusion and where the principal diffusivity is parallel to the streamlines the two-dimensional advection–diffusion equation for a spatially varying velocity field,

$$U(x,y)\frac{\partial C}{\partial x}+V(x,y)\frac{\partial C}{\partial y}=\frac{\partial}{\partial x}\left(K_{11}\frac{\partial C}{\partial x}+K_{12}\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial y}\left(K_{12}\frac{\partial C}{\partial x}+K_{22}\frac{\partial C}{\partial y}\right),$$

where

$$K_{11}=K_S\cos^2\theta+K_N\sin^2\theta,$$
$$K_{12}=(K_S-K_N)\sin\theta\cos\theta,$$
$$K_{22}=K_S\sin^2\theta+K_N\cos^2\theta,$$

may be transformed to (2.1)

$$\frac{\partial C}{\partial \Phi}=K_S\frac{\partial^2 C}{\partial \Phi^2}+K_N\frac{\partial^2 C}{\partial \Psi^2}\quad\text{for } K_S \text{ and } K_N \text{ constant,}$$

provided

$$U(x,y)=\frac{\partial \Phi}{\partial x}=\frac{\partial \Psi}{\partial y},$$
$$V(x,y)=\frac{\partial \Phi}{\partial y}=-\frac{\partial \Psi}{\partial x}.$$

Here θ is defined as the angle between the streamline and the x axis.

4.1. *Unequal flows*

In the analysis of §3 it was assumed that the speed of the flow, U, in each of the separate channels is equal. We may relax this restriction by the use of the above transformation. As an example for consideration the complex potential, $\Phi+i\Psi$, in the region of the junction of two channels of flows U_{I} and U_{II}, but of equal widths, b, is given by Churchill (1948),

$$\Phi(x^1,y^1)+i\Psi(x^1,y^1)=2(x^1+iy^1)+\frac{U_{\mathrm{I}}-U_{\mathrm{II}}}{\pi(U_{\mathrm{I}}+U_{\mathrm{II}})}\log\left(\frac{(1+\exp[-2\pi(-x^1+iy^1)])^{\frac{1}{2}}-1}{(1+\exp[-2\pi(-x^1+iy^1)])^{\frac{1}{2}}+1}\right). \quad (4.1)$$

The co-ordinates x^1, y^1 are scaled by b and the complex potential by $b(U_{\mathrm{I}}+U_{\mathrm{II}})$.

4.2. *Non-zero junction angle*

The method of conformal mapping provides a means for calculating the flow distribution in channels meeting at angles other than zero considered in §3. In the extreme case of a junction angle of 180°, that is, of a straight channel of width b, with opposed flow in each end and discharging through a slit in the channel wall, the non-dimensional complex potential is

$$\Phi(x^1,y^1)+i\Psi(x^1,y^1)=\frac{1}{\pi}\left\{\log\left(\exp[\pi(x^1+iy^1)]-1\right)-\frac{U_{\mathrm{II}}\pi}{U_{\mathrm{I}}+U_{\mathrm{II}}}(x^1+iy^1)\right.$$
$$\left.-\log\left[\left(\frac{U_{\mathrm{II}}}{U_{\mathrm{I}}}\right)^{U_{\mathrm{I}}/(U_{\mathrm{I}}+U_{\mathrm{II}})}+\left(\frac{U_{\mathrm{II}}}{U_{\mathrm{I}}}\right)^{-U_{\mathrm{II}}/(U_{\mathrm{I}}+U_{\mathrm{II}})}\right]\right\}.$$

The spatially invariant term in this expression ensures that the potential is zero at the stagnation point.

Conformal transformations for channels bent at arbitrary angles are given by Kober (1952, p. 156) which may be applied to junction angles other than those considered here.

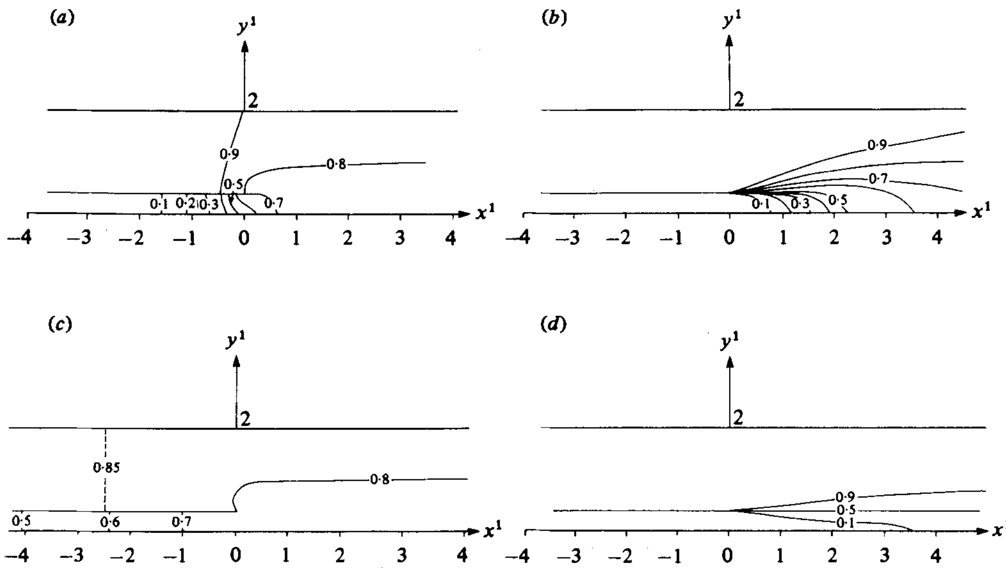

FIGURE 2. Distribution of concentration in channels of widths 0·4b and 1·6b. (a) $L_x = b$; (b) $L_x = 0{\cdot}1b$; (c) $L_x = 10b$; (d) $L_x = 0{\cdot}01b$ where $L_x = K_s/U = K_N/U$ and $C_{\rm I} \to 1$, $C_{\rm II} \to 0$, $x \to -\infty$.

5. Discussion

Figure 2 gives plots of $C(x,y)$ computed from (3.16) for a range of non-dimensional length scales, L_x/b, and for K_S equal to K_N. When advective and diffusive effects are balanced (figure 2a) the mixing zone of length scale L_x is equally distributed about the channel junction. In the case where diffusive effects predominate over advective effects or $L_x/b \gg 1$ the spatially dependent terms in (3·16c, d) are small and the first term in (3.16a, b) is larger than the second term. Therefore, the appropriate scale size for the upstream extent of the mixing zone is L_x. On the other hand, when advection is much stronger than diffusion or $L_x/b \ll 1$ the first two terms in (3.16a, b) cancel one another while the coefficient of x in the argument of the exponential of the first term in the sum in (3.16c, d) is approximately $(\pi/2b)^2 K_N/U$. Thus, the mixing zone characteristically extends from the junction to a distance, $(2b/\pi)^2\, U/K_N$, downstream of the junction. The distribution of figure 2(d) for very large advection is closely approximated by the asymptotic solution (3.17) in the region where the two flows meet. At a distance of several channel widths downstream the confining influence of the channel walls begins to be felt. Thus it is clear that the approximation, made by Sayre (1973), that the concentration at the junction of each of the two flows is unaltered from their upstream values is valid only in the advection-dominated case.

Figure 3 presents an example of the extension of the method to non-uniform velocity fields. The streamlines plotted in figure 3(a) and the associated velocity potential are mapped by the transformation (4.1) to a uniform field similar to that of figure (1). Equation (3.16) is then applied to the transformed flow field. The distribution of figure 3(b) is qualitatively similar to that of figure 2(a), which suggests that differing flow velocities play a similar role to that of differing channel widths in determining the characteristics of the mixing zone.

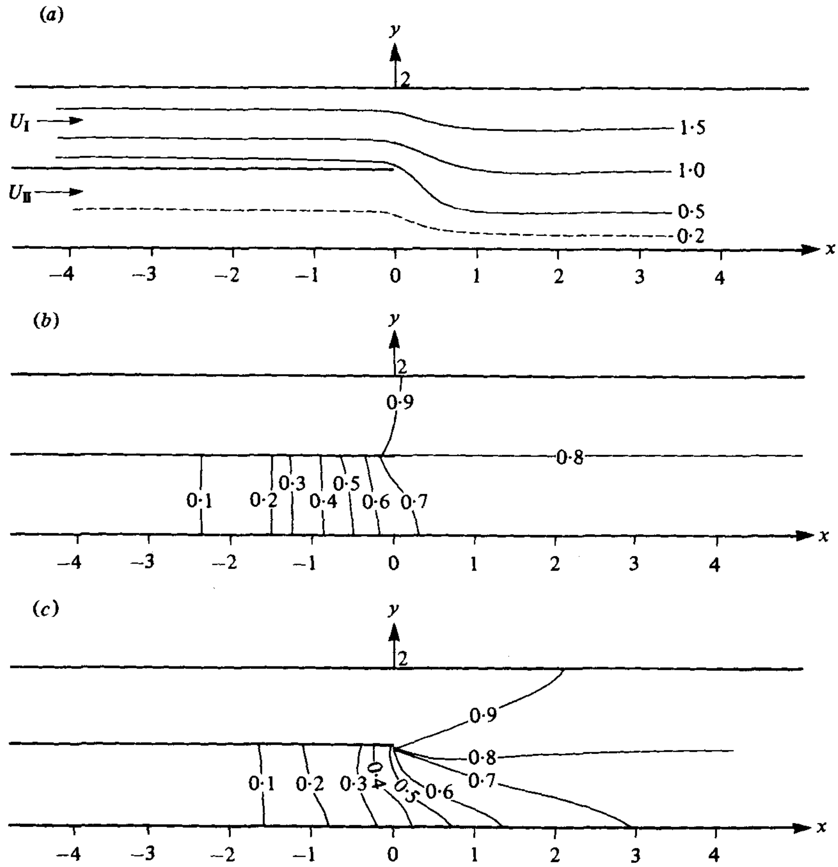

FIGURE 3. (a) Streamlines in two channels of equal widths, b, and $U_I/U_{II} = 4$. (b) Associated distribution of concentration for $K_s = K_N = b(U_I + U_{II})$. (c) Similarly for $K_s = 10K_N = b(U_I + U_{II})$. In (b) and (c) $C_I \to 1$, $C_{II} \to 0$ as $x \to -\infty$.

In a three-dimensional channel in which the flow is sheared in the vertical it is well known that the effect of the vertical shear is to augment the horizontal diffusion of the vertically averaged field of concentration. As an example of how the present theory may be applied to the problem of horizontal turbulent dispersion in a channel we assume that the augmented diffusion or dispersion coefficient in the direction parallel to the streamline is one order of magnitude larger than the diffusivity in the normal direction. By comparison of the resulting distribution shown in figure 3(c) with the case of isotropic diffusion, figure 3(b), it is concluded that the magnitude of the horizontal mixing zone is sensitive to lateral diffusion.

Finally, the flow in two channels meeting at the extreme junction angle of 180° is considered in figure 4(a). When advection is stronger than diffusion in both channels a frontal zone is found in the vicinity of the convergence of the two flows. The front is more pronounced on the side of the outflow. When advection is reduced the distribution is more diffuse, particularly in the direction of the weaker flow, figure 4(b).

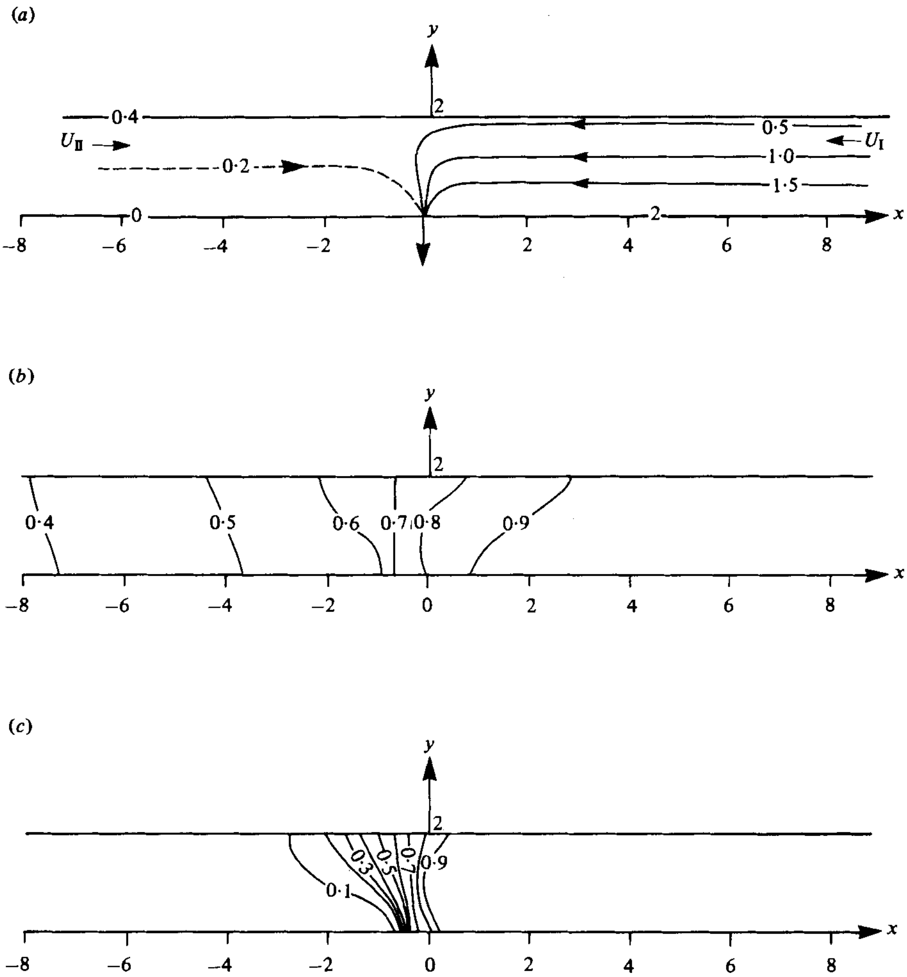

FIGURE 4. (a) Streamlines in a straight channel with the outflow at $x = 0$, $y = 0$ for $U_I/U_{II} = 4$. (b) Associated distribution of concentration for $K_S = K_N = b(U_I + U_{II})$. (c) Similarly for

$$K_S = K_N = 0{\cdot}1b(U_I + U_{II}).$$

In (b) and (c) $C_I \to 1$ as $x \to \infty$; $C_{II} \to 0$ as $x \to -\infty$.

6. Conclusion

Exact solutions of the steady advection–diffusion equation for a branching channel have been constructed by using the Wiener–Hopf technique. The theory given here can be applied to more realistic problems that might arise in practice, namely more general channel geometry, spatially varying flow fields and shear-augmented dispersion.

Of considerable interest is the problem of diffusion of a substance through a network of branching channels. In the treatment of such hierarchical systems assumptions are often made that the concentration at the junction is either completely mixed or unmixed in the lateral direction. The present theory supports such assumptions providing the individual compartments of the network have lengths greater than the appropriate diffusion length scales, L_x, or b^2L_x/L_y^2, for mixing to lateral uniformity.

Application of the present theory to a number of problems of practical interest in lakes, estuaries, and rivers is in progress.

Appendix

The theory of the decomposition of the function $K(\alpha)$ in terms of an infinite-product expression is found in Noble (1958). Following this procedure $K(\alpha)$ may be written as

$$K(\alpha) = \frac{c(2b-c)\gamma \sinh \gamma 2b}{2b \sinh \gamma c \sinh \gamma (b-c)} = K_+(\alpha)\, K_-(\alpha),$$

where

$$K_+(i\alpha) = \exp\left(\frac{\alpha}{\pi}(2b \ln 2b - c \ln c - (2b-c)\ln(2b-c))\right)$$
$$\times \prod_{n=1}^{\infty} \frac{\left\{\left(1+\left(\frac{b}{n\pi}\right)^2\right)^{\frac{1}{2}} + \alpha\frac{2b}{n\pi}\right\}}{\left\{\left(1+\left(\frac{c}{2n\pi}\right)^2\right)^{\frac{1}{2}} + \alpha\frac{c}{n\pi}\right\}\left\{\left(1+\left(\frac{2b-c}{2n\pi}\right)^2\right)^{\frac{1}{2}} + \alpha\frac{2b-c}{n\pi}\right\}}. \quad \text{(A 1)}$$

Since the $K(\alpha)$ is an even function of α, $K_+(i\alpha) = K_-(-i\alpha)$. In computation of $K_+(i\alpha)$ in the advection-dominated case, that is, b, c and $2b-c$ large, $K_+(i\alpha)$ may be approximated by

$$K_+(i\alpha) \simeq \left[\frac{c(2b-c)}{b}(\alpha+\tfrac{1}{2})\right]^{\frac{1}{2}}.$$

In the diffusion-dominated range, that is b, c, $2b-c$ small, the asymptotic value of $K_+(i\alpha)$, for large α, may be estimated from gamma-function formulae. In turn from Stirling's formula for the asymptotic form for the gamma function,

$$K_+(i\alpha) \simeq \frac{(2\pi)^{\frac{1}{2}}}{e}\alpha^{\frac{1}{2}} \quad \text{for} \quad \alpha \to \infty.$$

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